$$\Delta P_{\varkappa\lambda}+\frac{1}{1+\nu}\nabla_{\varkappa}\nabla_{\lambda}J_1+\frac{\nu}{1-\nu}\nabla^{\alpha}f_{\alpha}g_{\varkappa\lambda}+\nabla_{\varkappa}f_{\lambda}+\nabla_{\lambda}f_{\varkappa}=0 \qquad (3.9)$$

These are the Beltrami-Michell equations of compatibility in terms of stresses which are presented in the linear theory of elasticity [4, 5]. Raising the scripts $\varkappa$ and λ, we obtain the equations in the contravariant stress components

$$\Delta P^{\varkappa\lambda}+\frac{1}{1+\nu}\nabla^{\varkappa}\nabla^{\lambda}J_1+\frac{\nu}{1-\nu}\nabla^{\alpha}f_{\alpha}g^{\varkappa\lambda}+\nabla^{\varkappa}f^{\lambda}+\nabla^{\lambda}f^{\varkappa}=0 \qquad (3.10)$$

In Cartesian coordinates, Eqs. (3.9) and (3.10) have the form

$$\frac{\partial^2 P_{\varkappa\lambda}}{\partial x^{\alpha}\partial x^{\alpha}}+\frac{1}{1+\nu}\frac{\partial^2 J_1}{\partial x^{\varkappa}\partial x^{\lambda}}+\frac{\nu}{1-\nu}\frac{\partial f_{\alpha}}{\partial x^{\alpha}}\delta_{\varkappa\lambda}+\frac{\partial f_{\lambda}}{\partial x^{\varkappa}}+\frac{\partial f_{\varkappa}}{\partial x^{\lambda}}=0$$

Thus, the Beltrami-Michell equations of compatibility in terms of stresses correspond to geometrically and physically linear elasticity; Eqs. (3.6) are the generalizations of these equations in the case of geometric nonlinearity.

BIBLIOGRAPHY

1. Sedov, L. I., Introduction to the Mechanics of a Continuous Medium. (English Transl.), Addison-Wesley, Reading, Mass., 1965.
2. Sneddon, I. N. and Berry, D. S., The Classical Theory of Elasticity, Handbuch der Physik, Bd. VI, Springer, Berlin, 1958.
3. Novozhilov, V. V., Theory of Elasticity (English Transl.), Israel Program for Scientific Translations, Jerusalem, 1961.
4. Germain, P., Mécanique des Milieux Continus. Masson, Paris, 1962.
5. Muskhelishvili, N. I., Some Basic Problems of the Mathematical Theory of Elasticity (English Transl. of the 3rd Ed.), Noordhoff, Groningen, Holland, 1953.

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ON FUNDAMENTAL SOLUTIONS IN THE THEORY OF CIRCULAR CYLINDRICAL SHELLS

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The problem of a circular cylindrical shell of elastic isotropic material subjected to concentrated loadings is considered. As is known, such a problem in two-dimensional formulation (based on Kirchhoff-Love hypotheses) reduces to the construction of the Green's function for an elliptic equation in the resolution function.

A fundamental solution in closed form has been obtained in [1, 2] for the shallow cylindrical shell equations by using Fourier transforms. A method of the theory of generalized functions [4] was utilized in [3] to construct a fundamental solution of the equations of the theory of shells of positive Gaussian curvature.

Fundamental solutions are constructed below for the most prevalent modifications of the theory of nonshallow circular cylindrical shells [5 – 8]. In contrast to [1 – 3], the "classical" method of plane waves and spherical means [9] is utilized which permits, so to say, construction of the fundamental solution for an elliptic operator with constant coefficient to any accuracy by algorithmic and elementary means.

A qualitative analysis of the fundamental solutions is given. A method is presented for strengthening the convergence of the periodic fundamental solution. The error in a representation associated with an analysis of concentrated couple loading is noted. The question of the limits of applicability of applied shell theory to the analysis of local loadings is not touched upon.

1. A fundamental solution is constructed for an operator of the form

$$\Delta^4 + f\frac{\partial^6}{\partial x^6} + g\frac{\partial^6}{\partial x^4\partial\beta^2} + l\frac{\partial^6}{\partial x^2\partial\beta^4} + 2\frac{\partial^6}{\partial\beta^6} + \tag{1.1}$$

$$+\frac{1-\sigma^2}{a^2}\frac{\partial^4}{\partial\alpha^4} + v\frac{\partial^4}{\partial x^2\partial\beta^2} + \frac{\partial^4}{\partial\beta^4} \qquad \left(a^2 = \frac{h^2}{3R^2}\right)$$

Here Δ is the Laplace operator, R is the radius, h is half the shell thickness, α, β are the dimensionless longitudinal and transverse coordinates, σ is the Poisson's ratio.

The operator (1.1) corresponds to the considered modifications of the theory of circular cylindrical shells for the following values of the coefficients:

	[5]	[6]	[7]	[8]
$f=$	2σ	0	0	0
$g=$	6	$8-2\sigma^2$	$7-\sigma^2$	2
$l=$	$8-2\sigma$	8	8	4
$v=$	2	4	4	0

According to [9], the singular (Φ_1) and regular (Φ_2) parts of the fundamental solution Φ ($\Phi = \Phi_1 + \Phi_2$) are for even n and $n < m$:

$$(2\pi i)^n \Phi_1(x, y) = -(\Delta_y)^{1/2 n}\int_\Omega \nu(x, \xi, y\xi)\ln|(x-y)\xi|\,d\omega \tag{1.2}$$

$$(2\pi i)^n \Phi_2(x, y) = -(\Delta_y)^{1/2 n}\int_\Omega d\omega \int_0^{(x-y)\xi} \frac{\nu(x, \xi, t+y\xi) - \nu(x, \xi, y\xi)}{t}\,dt \tag{1.3}$$

$$\nu(x, \xi, p) = \frac{1}{2\pi i}\int_C \frac{e^{(x\xi-p)\lambda}}{\lambda P(\lambda\xi)}\,d\lambda \tag{1.4}$$

Here m is the order of the operator, $x\ (x_1,\ldots,x_n)$ is a vector in n space, y is a fixed vector from the origin to the singular point, $P(\lambda\xi)$ is a polynomial obtained from the elliptic operator by symbolic replacement of differentiation with respect to $x_1,\ldots,x_n$ by multiplication by $\lambda\xi_1,\ldots,\lambda\xi_n$; $x\xi$ is the scalar product, the subscript Ω on the integral means integration over a unit radius sphere in the space $\{\xi\}$ and integration with respect to λ is over the closed contour C in the complex λ plane, which encloses all the roots of $\lambda P(\lambda\xi)$.

By simple manipulation we obtain for $n = 2$

$$\Delta_y[\nu\ln|(x-y)\xi|] = \left[\frac{\partial^2\nu}{\partial p^2}\ln|x\xi - p| - 2\frac{\partial\nu}{\partial p}\frac{1}{(x\xi-p)} - \frac{\nu}{(x\xi-p)^2}\right]_{p=y\xi} \tag{1.5}$$

$$\Delta_y\nu = \left(\frac{\partial^2\nu}{\partial p^2}\right)_{p=y\xi}, \quad \left(\frac{\partial^2\nu}{\partial p^2}\right)_{p=y\xi} = \frac{1}{2\pi i}\int_C \frac{e^{(x-y)\xi\lambda}\lambda}{P(\lambda\xi)}\,d\lambda \tag{1.6}$$

$$\left(\frac{\partial\nu}{\partial p}\right)_{p=y\xi} = -\frac{1}{2\pi i}\int_C \frac{e^{(x-y)\xi\lambda}}{P(\lambda\xi)}\,d\lambda \tag{1.7}$$

Substituting (1. 5) into (1. 2) and (1. 6) into (1. 3) we obtain

$$-(2\pi i)^2 \Phi_1(x, y) = \frac{1}{2\pi i}\int_\Omega \ln|(x-y)\xi|\,d\omega \int_c \frac{e^{(x-y)\xi\lambda}\lambda}{P(\lambda\xi)}\,d\lambda +$$

$$+\frac{1}{\pi i}\int_\Omega \frac{d\omega}{(x-y)\xi}\int_c \frac{e^{(x-y)\xi\lambda}}{P(\lambda\xi)}\,d\lambda - \frac{1}{2\pi i}\int_\Omega \frac{d\omega}{[(x-y)\xi]^2}\int_c \frac{e^{(x-y)\xi\lambda}}{\lambda P(\lambda\xi)}\,d\lambda \tag{1.8}$$

$$-(2\pi i)^2 \Phi_2(x,y) = \frac{1}{2\pi i}\int_\Omega d\omega \int_0^{(x-y)\xi} \frac{dt}{t}\int_0 \frac{e^{(x-y)\xi\lambda - t\lambda} - e^{(x-y)\xi\lambda}}{P(\lambda\xi)}\lambda\,d\lambda$$

Let us note that the second and third members of Φ_1 are regular. The polynomial $P(\lambda\xi)$ has the following form in the problem under consideration:

$$P(\lambda\xi) = \lambda^8 + (f\xi_1^6 + g\xi_1^4\xi_2^2 + l\xi_1^2\xi_2^4 + 2\xi_2^6)\lambda^6 + \\ + [(1-\sigma^2)a^{-2}\xi_1^4 + v\xi_1^2\xi_2^2 + \xi_2^4]\lambda^4 \tag{1.9}$$

Formula (1. 9) is written taking into account the relationship $\xi_1^2 + \xi_2^2 = 1$ since the polynomial $(\lambda\xi)$ will be utilized on the unit circle.

Let us integrate with respect to λ in (1. 8) by using residues and let us expand the exponentials hence obtained in Taylor series; we have as a result

$$4\pi^2\Phi_1(x, y) = \int_\Omega \ln|(x-y)\xi|\left\{\frac{[(x-y)\xi]^6}{6!} + \frac{[(x-y)\xi]^8}{8!}(\lambda_1^2+\lambda_2^2) + \right.$$

$$\left. + \frac{[(x-y)\xi]^{10}}{10!}(\lambda_1^4 + \lambda_1^2\lambda_2^2 + \lambda_2^4) + \frac{[(x-y)\xi]^{12}}{12!}(\lambda_1^6 + \lambda_1^4\lambda_2^2 + \lambda_1^2\lambda_2^4 + \lambda_2^6) + \cdots\right\}d\omega +$$

$$+2\int_\Omega \left\{\frac{[(x-y)\xi]^6}{7!} + \frac{[(x-y)\xi]^8}{9!}(\lambda_1^2+\lambda_2^2) + \cdots\right\}d\omega -$$

$$-\int_\Omega \left\{\frac{[(x-y)\xi]^6}{8!} + \frac{[(x-y)\xi]^6}{10!}(\lambda_1^2+\lambda_2^2) + \cdots\right\}d\omega \tag{1.10}$$

$$4\pi^2\Phi_2(x, y) = \int_\Omega \left\{\int_0^1 \frac{(1-t)^6-1}{t}\,dt\,\frac{[(x-y)\xi]^6}{6!} + \right.$$

$$\left. + \int_0^1 \frac{(1-t)^8-1}{t}\,dt\,\frac{[(x-y)\xi]^8}{8!}(\lambda_1^2+\lambda_2^2) + \cdots\right\}d\omega \tag{1.11}$$

Here λ_1, λ_2 are nonzero roots of $P(\lambda\xi)$.

Transforming to a polar dimensionless coordinate system (r, φ) with origin at the singular point, we convert (1. 10), (1. 11) into

$$4\pi^2\Phi_1(r, \varphi) = \ln r\left[\frac{r^6}{6!}\int_0^{2\pi}\cos^6(\varphi-\eta)\,d\eta + \right. \tag{1.12}$$

$$\left. + \frac{r^8}{8!}\int_0^{2\pi}\cos^8(\varphi-\eta)(\lambda_1^2+\lambda_2^2)\,d\eta + \cdots\right] + \frac{r^6}{6!}\int_0^{2\pi}\ln|\cos(\varphi-\eta)|\cos^6(\varphi-\eta)\,d\eta +$$

$$+\frac{r^8}{8!}\int_0^{2\pi}\ln|\cos(\varphi-\eta)|\cos^8(\varphi-\eta)(\lambda_1^2+\lambda_2^2)\,d\eta + \cdots +$$

$$+\left(\frac{2}{7!}-\frac{1}{8!}\right)r^6\int_0^{2\pi}\cos^6(\varphi-\eta)\,d\eta + \left(\frac{2}{9!}-\frac{1}{10!}\right)r^8\int_0^{2\pi}\cos^8(\varphi-\eta)(\lambda_1^2+\lambda_2^2)\,d\eta + \cdots$$

$$4\pi^2\Phi_2(r,\varphi)=\frac{r^6}{6!}\int_0^1\frac{(1-t)^6-1}{t}\,dt\int_0^{2\pi}\cos^6(\varphi-\eta)\,d\eta+$$
$$+\frac{r^8}{8!}\int_0^1\frac{(1-t)^8-1}{t}\,dt\int_0^{2\pi}\cos^8(\varphi-\eta)\,(\lambda_1^2+\lambda_2^2)\,d\eta+\cdots$$

Hence

$$\lambda_1^2+\lambda_2^2=-f\cos^6\eta-g\cos^4\eta\sin^2\eta-l\cos^2\eta\sin^4\eta-2\sin^6\eta \tag{1.13}$$

$\lambda_1^4+\lambda_1^2\lambda_2^2+\lambda_2^4=(\lambda_1^2+\lambda_2^2)^2-\lambda_1^2\lambda_2^2,\ \lambda_1^2\lambda_2^2=(1-\sigma^2)\,a^{-2}\cos^4\eta+v\cos^2\eta\sin^2\eta+\sin^4\eta$ etc.

2. Let us examine certain qualitative questions. Estimating the integrals in (1.12), we see that among the terms in the polynomial $P(\lambda\xi)$ which correspond to the fourth derivative, $(1-\sigma^2)a^{-2}\cos^4\eta$ plays a predominant part, while the rest can be neglected (for not very long shells, [10], p. 544) during the process of constructing the fundamental solution. In general, only the member with the coefficient $(1-\sigma^2)a^{-2}$ should remain from any sum in the integrand. For example, taking the integrals

$$\int_0^{2\pi}\cos^{10}(\varphi-\eta)(\lambda_1^4+\lambda_1^2\lambda_2^2+\lambda_2^4)\,d\eta,\qquad \int_0^{2\pi}\ln|\cos(\varphi-\eta)|\cos^{10}(\varphi-\eta)(\lambda_1^4+\lambda_1^2\lambda_2^2+\lambda_2^4)\,d\eta$$

we can put $(\lambda_1^4+\lambda_1^2\lambda_2^2+\lambda_2^4)$ equal to $(1-\sigma^2)\,a^{-2}\cos^4\eta$ for all the operators investigated, which simplifies the calculation considerably.

It is easy to see that for the odd members of the singular and regular series the simplification noted is equivalent to calculating them by shallow shell theory.

Table 1

	[5]	[6]	[7]	[8]
s_1	$5.12-1.12\sigma$	$21.62-4.12\sigma^2$	$19.56-2.06\sigma^2$	7
s_2	$82.38-16.38\sigma$	$191.88-27.38\sigma^2$	$178.19-13.69\sigma^2$	77
s_3	$171.38-27.33\sigma$	$236.88-16.33\sigma^2$	$228.69-8.19\sigma^2$	133
s_4	$70.12-4.12\sigma$	$74.62-1.12\sigma^2$	$74.06-0.57\sigma^2$	63
s_5	107.25	107.25	107.25	107.25
s_6	49.50	49.50	49.50	49.50
s_7	2.25	2.25	2.25	2.25
γ_1	$46.06+132.34\sigma$	$56.37-10.37\sigma^2$	$51.00-0.70\sigma^2$	3.60
γ_2	$416.16+119.06\sigma$	$483.44-67.61\sigma^2$	$449.53-33.85\sigma^2$	32.60
γ_3	$557.48-22.27\sigma$	$599.33-41.90\sigma^2$	$578.43-20.95\sigma^2$	54.40
γ_4	$187.33-8.98\sigma$	$190.41-28.24\sigma^2$	$183.90-1.51\sigma^2$	25.40
γ_5	297.56	297.56	297.56	297.56
γ_6	140.96	140.96	140.96	140.96
γ_7	6.62	6.62	6.62	6.62

All the even numbers of the fundamental solution have the factor $(\lambda_1^2+\lambda_2^2)$ in the appropriate integrals, because of the presence of sixth derivatives. In this connection, it is pertinent to consider the following question. It is mentioned in [10] (p. 535, formula (2.8)) that the member $(7-\sigma^2)k^2$ can be neglected in the expression $6k^4-(7-\sigma^2)k^2+$ $+(1-\sigma^2)a^{-2}$ since it is small compared to $(1-\sigma^2)a^{-2}$ for small k and compared to $6k^4$ for large k (k is the number of the harmonic in the trigonometric series in the coordinate β). However, it is seen from (1.13) that the role of this member in the fundamental solution is the same as the remaining sixth derivatives, hence, if it were systematic, it would be necessary either to retain all sixth derivatives or to discard all.

Discarding the sixth derivatives and taking into account what has been said above about the fourth derivatives means a complete transition to shallow shell theory, hence all the even members of the fundamental solution vanish.

Let us present trinomial expressions of the singular and regular parts of the fundamental solution for the considered modifications of the theory of a circular cylindrical shell (the values of the coefficients $s_1, \ldots, s_7$, $\gamma_1, \ldots, \gamma_7$ are presented in Table 1)

$$\begin{aligned}\Phi(r, \varphi) = \frac{1}{\pi} \ln r \Big[\frac{r^6}{576 \cdot 8} - \frac{r^8}{8!\, 2^8} (s_1 \cos^6 \varphi + s_2 \cos^4 \varphi \sin^2 \varphi + s_3 \cos^2 \varphi \sin^4 \varphi + s_4 \sin^6 \varphi) - \\ - \frac{r^{10}}{10!\, 2^{10}} (1 - \sigma^2) a^{-2} (s_5 \cos^4 \varphi + s_6 \cos^2 \varphi \sin^2 \varphi + s_7 \sin^4 \varphi) \Big] - \\ - \frac{r^6 \cdot 22{,}6}{6!\, 2^6 \pi} + \frac{r^8}{8!\, 2^8 \pi} (\gamma_1 \cos^6 \varphi + \gamma_2 \cos^4 \varphi \sin^2 \varphi + \gamma_3 \cos^2 \varphi \sin^4 \varphi + \gamma_4 \sin^6 \varphi) + \\ + \frac{r^{10}}{10!\, 2^{10} \pi} (1 - \sigma^2) a^{-2} (\gamma_5 \cos^4 \varphi + \gamma_6 \cos^2 \varphi \sin^2 \varphi + \gamma_7 \sin^4 \varphi) + \cdots \qquad (2.1)\end{aligned}$$

Let us turn attention to the following fact. The internal bending moments have a logarithmic singularity under the effect of a normal concentrated force. Diverse fundamental solutions (defined to the accuracy of the regular solution of the homogeneous equation) yield diverse regular additions to the singular part, and since the logarithm is a weak singularity, these additions can be commensurate at distances on the order of h as is graphically illustrated by the asymptotic representation of the solution [2]

$$\ln r \sqrt{R/h} = \ln r + {}^1\!/_2 \ln (R/h)$$

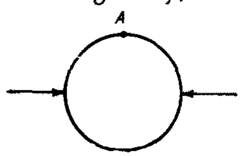

Fig. 1

There results from this example that the fundamental solutions are provisional criteria for estimating the strength of a shell even in a small neighborhood of the local action.

Let us expound some considerations apropos the fundamental solution [2]. The solution mentioned damps asymptotically in the transverse direction as

$$(\beta \sqrt{R/h})^{-0.5} e^{-\beta \sqrt{R/h}} \qquad (\alpha = 0, \quad \beta \to \infty)$$

which contradicts the physical picture since the zone A (Fig. 1) turns out to be free of stresses and strains in the bending of a cylindrical shell by two concentrated forces. The solution [2] also damps out in the longitudinal direction. This can apparently be explained [11] by the selection of the shallow shell theory equations (a solution having power gtowth in the longitudinal direction is obtained in [10, 11] for "non-shallow" shells). The question of why the solution [2] damps in the transverse direction remains unclear. Hence, rejecting the periodicity requirement can hardly be motivated by the fact that the solution damps rapidly in the circumferential direction (*). Moreover, the advantages of the solution [2] are reduced in the plane of an analytically correct description of the stress-strain state and its numerical estimation at some distance from the zone of local action.

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*) V. P. Shevchenko, Stress-strain state of shells in the neighborhood of concentrated loadings. Dissertation Abstract. Dnepropetrovsk State Univ., 1966.

3. Fundamental solutions are constructed in the complete argument space by methods applied herein and in [1–3]. Such solutions should probably be used to analize open cylindrical shells. the method of reinforcing the convergence, utilized in [10, 11] corresponds better to the nature of the problem in the case of a closed cylindrical shell. This method can be developed during the process to assure as significant an improvement in the convergence as desired. The computations are simplest if the Novozhilov equations [8] are used.

The solution in [10] is obtained in the form of the series

$$\Phi = \sum_{k=0}^{\infty} f_k(\alpha) \cos k\beta, \qquad f_k(\alpha) \sim \frac{1}{k^7} \int_{-\infty}^{\infty} \frac{e^{i\alpha k\eta}}{\Delta_k(\eta)} d\eta \tag{3.1}$$

$$\Delta_k(\eta) = [(\eta^2+1)^2 - k^{-2}]^2 + k^{-4}(1-\sigma^2)a^{-2}\eta^4 \tag{3.2}$$

Reinforcement of the convergence is achieved in [11] by the transformation

$$f_k(\alpha) \sim f_{k1}(\alpha) + N_{k1}(\alpha) \qquad (k > 1) \tag{3.3}$$

$$f_{k1}(\alpha) = \frac{1}{k^7} \int_{-\infty}^{\infty} \frac{e^{i\alpha k\eta}}{(1+\eta^2)^4} d\eta = \frac{2\pi}{3!\,2^4} \frac{1}{k^7} (k^3|\alpha|^3 + 6k^2\alpha^2 + 15k|\alpha| + 15) e^{-k|\alpha|} \tag{3.4}$$

$$N_{k1}(\alpha) = \frac{1}{k^7} \int_{-\infty}^{\infty} \left[\frac{2}{k^2}(1+\eta^2)^2 - \frac{1}{k^4}\left(1 + \frac{1-\sigma^2}{a^2}\eta^4\right)\right] \frac{e^{i\alpha k\eta}}{(1+\eta^2)^4 \Delta_k(\eta)} d\eta \tag{3.5}$$

with the subsequent summation of the series $\Sigma f_{k1}(\alpha)\cos k\beta$ in closed form; the series $\Sigma N_{k1}(\alpha)\cos k\beta$ converges more rapidly than $\Sigma f_k(\alpha)\cos k\beta$ which is easily established by the order of the decrease in its coefficients as k grows.

In its turn, the expression $N_{k1}(\alpha)$ can be represented as

$$N_{k1}(\alpha) = f_{k2}(\alpha) + N_{k2}(\alpha) \tag{3.6}$$

$$f_{k2}(\alpha) = \frac{2}{k^9} \int_{-\infty}^{\infty} \frac{e^{i\alpha k\eta}}{(1+\eta^2)^6} d\eta - \frac{1}{k^{11}} \int_{-\infty}^{\infty} \left(1 + \frac{1-\sigma^2}{a^2}\eta^4\right) \frac{e^{i\alpha k\eta}}{(1+\eta^2)^8} d\eta =$$
$$= \frac{4\pi}{2^6 5!\, k^9} (k^5|\alpha|^5 + 15k^4\alpha^4 + 105k^3|\alpha|^3 + 420k^2\alpha^2 + 945k|\alpha| + 945) e^{-k|\alpha|} -$$
$$- \frac{1}{k^{11}} \int_{-\infty}^{\infty} \left(1 + \frac{1-\sigma^2}{a^2}\eta^4\right) \frac{e^{i\alpha k\eta}}{(1+\eta^2)^8} d\eta \tag{3.7}$$

$$N_{k2}(\alpha) = \frac{1}{k^7} \int_{-\infty}^{\infty} \left[\frac{2}{k^2}(1+\eta^2)^2 - \frac{1}{k^4}\left(1 + \frac{1-\sigma^2}{a^2}\eta^4\right)\right]^2 \frac{e^{i\alpha k\eta}}{(1+\eta^2)^8 \Delta_k(\eta)} d\eta \tag{3.8}$$

etc. This process (with q number of steps) can be written as follows:

$$f_k(\alpha) \sim \frac{1}{k^7} \int_{-\infty}^{\infty} \frac{e^{i\alpha k\eta}}{\Delta_k(\eta)} d\eta = \tag{3.9}$$
$$= \frac{1}{k^7} \int_{-\infty}^{\infty} \frac{e^{i\alpha k\eta}}{(1+\eta^2)^4} \left\{1 - \frac{1}{(1+\eta^2)^4}\left[\frac{2}{k^2}(1+\eta^2)^2 - \frac{1}{k^4}\left(1 + \frac{1-\sigma^2}{a^2}\eta^4\right)\right]\right\}^{-1} d\eta =$$
$$= \frac{1}{k^7} \int_{-\infty}^{\infty} \frac{e^{i\alpha k\eta}}{(1+\eta^2)^4} \sum_{j=1}^{q} \left[\frac{2}{k^2}(1+\eta^2)^2 - \frac{1}{k^4}\left(1 + \frac{1-\sigma^2}{a^2}\eta^4\right)\right]^{j-1} \frac{d\eta}{(1+\eta^2)^{4(j-1)}} + R_{kq}(\alpha) =$$

$$= \sum_{j=1}^{q} f_{kj}(\alpha) + R_{kq}(\alpha)$$

Taking the integrals $f_{kj}(\alpha)$, we obtain after simple manipulation

$$\sum_{j=1}^{q} \sum_{k=2}^{\infty} f_{kj}(\alpha) \cos k\beta$$

as the sum of closed expressions and a series which converges as rapidly as the series with the common member $R_{kq}(\alpha) \cos k\beta$.

4. The erroneous view of the possibility of obtaining a fundamental function corresponding to a concentrated moment M by differentiating the fundamental solution of the case of the concentrated normal force is expressed in a number of papers.

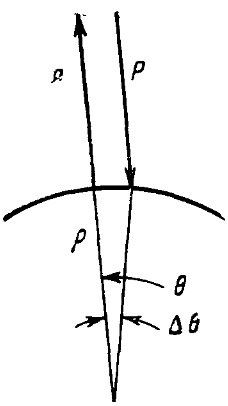

Fig. 2

Let us examine the force couple applied to the shell at points of the line of curvature with coordinates θ, $\theta - \Delta\theta$ (Fig. 2). Denoting the fundamental function of the normal and tangential concentrated force cases by $\Phi^*(\theta)$, $\Phi^{**}(\theta)$, we write the fundamental solution for the considerd force couple as

$$\Phi = P\Phi^*(\theta) - P \cos \Delta\theta\, \Phi^*(\theta - \Delta\theta) - P \sin \Delta\theta \Phi^{**}(\theta) \tag{4.1}$$

By definition

$$M = \lim \rho \Delta\theta P \qquad \text{for} \quad \Delta\theta \to 0, \ P \to \infty \tag{4.2}$$

Here ρ is the radius of the considered line of curvature. Performing the obvious manipulations, and passing to the limit in (4.1), we obtain

$$\lim \Phi = \frac{M}{\rho}\left(\frac{\partial \Phi^*}{\partial \theta} - \Phi^{**}\right), \qquad \Delta\theta \to 0, \qquad P \to \infty \tag{4.3}$$

For a cylindrical shell the mentioned correction must be taken into account in computing the transverse bending moment. It equals zero in the case of the longitudinal bending moment.

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BIBLIOGRAPHY

1. Shevliakov, Iu. A. and Shevchenko, V. P., On the problem of the effect of concentrated loadings on a shallow shell. "Stress Concentrations", Kiev, "Naukova Dumka", №1, 1965.
2. Shevliakov, Iu. A. and Shevchenko, V. P., On the effect of concentrated forces and bending moments on a shallow cylindrical shell. Prikl. Mekh. Vol. 2, №1, 1966.
3. Chernyshev, G. N., Asymptotic methods in shell theory (concentrated loads). Trudy VI All-Union Conference on Plate and Shell Theory. Moscow, "Nauka", 1966.
4. Gel'fand, I. M. and Shilov, G. E., Generalized Functions and Operations on Them. Moscow, Fizmatgiz, 1958.
5. Vlasov, V. Z., General Shell Theory and its Application in Engineering. Moscow-Leningrad, Gostekhizdat, 1949.
6. Gol'denveizer, A.L., Theory of Elastic Thin Shells, Moscow, Gostekhizdat, 1953.

7. Love, A., Mathematical Theory of Elasticity (Russian translation). Moscow-Leningrad, ONTI, 1935.
8. Novozhilov, V. V., Theory of Thin Shells, 2nd Ed., Leningrad, Sudpromgiz, 1962.
9. John, F., Plane Waves and Spherical Means Applied to Partial Differential Equations. N.Y., Interscience, 1958.
10. Darevskii, V. M., On the theory of cylindrical shells. PMM Vol.15, №5, 1951.
11. Darevskii, V. M., Solution of some problems of the theory of the cylindrical shell. PMM Vol.16, №2, 1952.
12. Vorovich, I. I., Safronov, Iu. V. and Ustinov, Iu. A., Strength of Wheels of Complex Construction. Research and Analysis. Moscow, Mashinostroenie, 1967.

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RESONANCE OSCILLATIONS OF A SPECIAL DOUBLE PENDULUM

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Resonance oscillations of a mechanical system are investigated, and peculiarities in its behavior are explained. The oscillations of conservative systems with two degrees of freedom under internal resonance are examined in [1−5]. A certain addition to the existing asymptotic methods in the theory of nonlinear oscillations is proposed in the last paper by Struble; the results of this paper are utilized below.

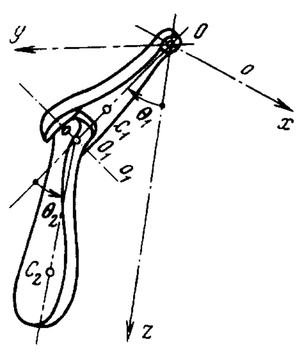

Fig. 1

1. Let us consider a system of two successively connected physical pendulums (Fig. 1). The first rotates around a horizontal axis o, and the second around an axis o_1 belonging to the first pendulum and perpendicular to o. In the equilibrium position o_1 is horizontal. Let C_1 and C_2 denote the centers of gravity of the two physical pendulums; M_1 and M_2 their masses; O_1 the intersection of the line OC_1 with the o_1-axis; I_1 the moment of inertia of the first pendulum relative to an axis passing through C_1 and parallel to o; I_{22} the moment of inertia of the second pendulum relative to the axis passing through C_2 and parallel to o_1; I_{23} passing through C_2 and O_1; I_{21} passing through C_2 and perpendicular to the other two axes. We shall consider I_{21}, I_{22} and I_{23} to be the principal central moments of inertia of the second pendulum; let θ_1 be the deflection of the first pendulum from the oz-axis, and θ_2 the deflection of the second pendulum from the OC_1-axis; let us set

$$OC_1 = a_1,\ O_1C_2 = a_2,\ OO_1 = b_1$$

In this notation we have:

for the system kinetic energy